# Analysing the master equation for qubits

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According to the postulates of quantum mechanics, the evolution of a closed system is in the form of a unitary. But when we study open quantum systems, we have to generalise the notion of a measurement and evolution to account for the interaction between the system and the environment. If we consider the closed system of the system under consideration and the environment (a philosophical problem, but let's ignore that for while), we can consider the system and the environment as the complete system, and study it using a unitary evolution. But since we care about the system and not the environment, we trace out the environment after the complete evolution, and study the system we want. This can alternately be thought of as an application of a general map $\varepsilon_{(t_1,t_0)}$ that evolves the system under consideration, accounting for the interaction with the environment. It is depicted in the figure 1 below. More on this here.[1]

$$\begin{tikzcd}
\rho_A(t_0) \otimes \rho_B(t_0) \arrow[r, "U(t_1, t_0)"] \arrow[d, "\mathrm{Tr}_B"'] & \rho_{AB}(t_1) \arrow[d, "\mathrm{Tr}_B"'] \\
\rho_A(t_0) \arrow[r, "\mathcal{E}_{(t_1, t_0)}"] & \rho_A(t_1)
\end{tikzcd}$$

Figure 1: General evolution of a quantum system

Here, we will present a master equation for a two-level quantum (qubit) system, and solve it to get the dynamical map $\Lambda$. We will then calculate the fixed state of the map. The, we will find the Kraus operators corresponding to the map. Finally, we will calculate the entropy production rate for the qubit, and try to analyse it in various ways.

In this first submission of the report, we will do the first two parts, and introduce the maser equation. In the second and final submission, we will solve the other two parts as well, and analyse the work.

## 1 The Master Equation

Here we will study a system with markovian approximation. The Lindblad (or Gorini-Kossakowski-Sudarshan-Lindblad) master equation plays a key role in this study as it is the most general generator of Markovian dynamics in quantum systems[2]. The equation we want to study is the following:

$$\frac{d\rho}{dt} = L_1(\rho) + L_2(\rho) \tag{1}$$

where,

$$L_i = \gamma_i(n_i+1)(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}) + \gamma_i n_i(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}) \tag{2}$$

## 2 The Dynamical Map

Let's say that the initial density matrix is given as follows:

$$\rho(0) = \begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} \tag{3}$$

And at time $t$, the density matrix is:

$$\rho(t) = \begin{pmatrix} a(t) & b(t) \\ b^*(t) & c(t) \end{pmatrix} = \Lambda(t)\rho(0) \tag{4}$$

$$\frac{d\rho}{dt} = \begin{pmatrix} \dot{\mathrm{a}}(\mathrm{t}) & \dot{\mathrm{b}}(\mathrm{t}) \\ \dot{\mathrm{b}}^*(\mathrm{t}) & \dot{\mathrm{c}}(\mathrm{t}) \end{pmatrix} \tag{5}$$

We want to find the final density matrix in terms of the initial matrix. We know the following identities about the pauli matries:

$$\sigma_0 = \mathbb{I}_2, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_\pm = \sigma_x \pm i \cdot \sigma_y, \sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Therefore,

$$\sigma_+ = 2\left|0\right\rangle\left\langle 1\right|, \sigma_- = 2\left|1\right\rangle\left\langle 0\right|$$

Now, given

$$L_i\rho = \underbrace{\gamma_i(n_i+1)}_{\Gamma_i}\underbrace{(\sigma_-\rho\sigma_+ - \frac{1}{2}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-))}_{X} + \underbrace{\gamma_i n_i}_{\Delta_i}\underbrace{(\sigma_+\rho\sigma_- - \frac{1}{2}(\sigma_-\sigma_+\rho + \rho\sigma_-\sigma_+))}_{Y} \tag{6}$$

Let's simplify.

$$X = 4\left|1\right\rangle \left\langle 0|\rho|0\right\rangle \left\langle 1\right| - \frac{1}{2}(4\left|0\right\rangle \left\langle 0|\rho + 4\rho|0\right\rangle \left\langle 0\right|$$
$$\implies X = 4\begin{pmatrix} 0 & 0 \\ 0 & a(t) \end{pmatrix} - 2(\begin{pmatrix} a(t) & b(t) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a(t) & 0 \\ b^*(t) & 0 \end{pmatrix})$$
$$\implies X = \begin{pmatrix} -4a(t) & -2b(t) \\ -2b^*(t) & 4a(t) \end{pmatrix}$$

$$Y = 4\left|0\right\rangle \left\langle 1|\rho|1\right\rangle \left\langle 0\right| - \frac{1}{2}\cdot 2(\left|1\right\rangle \left\langle 1|\rho + \rho|1\right\rangle \left\langle 1\right|)$$
$$Y = \begin{pmatrix} 4c(t) & 0 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 0 & b(t) \\ b^*(t) & 2c(t) \end{pmatrix}$$
$$\implies Y = \begin{pmatrix} 4c(t) & -2b(t) \\ -2b^*(t) & -4c(t) \end{pmatrix}$$

$$\frac{d\rho}{dt} = (\Gamma_1 + \Gamma_2)X + (\Delta_1 + \Delta_2)Y$$
$$\text{Also, } a(t) + c(t) = 1 \ (\because Tr(\rho) = 1)$$
$$\therefore c(t) = 1 - a(t)$$
$$\frac{d\rho}{dt} = (\Gamma_1 + \Gamma_2)\begin{pmatrix} -4a & -2b \\ -2b^* & 4a \end{pmatrix} + (\Delta_1 + \Delta_2)\begin{pmatrix} 4(1-a) & -2b \\ -2b^* & -4(1-a) \end{pmatrix}$$

Comparing with equation (5), we get 2 differential equations. Solving each of them, and using the trace preservation property:

$$\frac{da(t)}{dt} = (-4)a(t)(\Gamma_1 + \Gamma_2) + (\Delta_1 + \Delta_2)(-4)(1 - a(t))$$
$$\frac{da(t)}{dt} = 4[-a(t)((\Gamma_1 + \Gamma_2 + \Delta_1 + \Delta_2) + \Delta_1 + \Delta_2]$$

Let $\Gamma = \Gamma_1 + \Gamma_2, \Delta = \Delta_1 + \Delta_2$. Substituting,

$$\frac{da}{dt} = 4(\Delta - a(\Gamma + \Delta))$$
$$\int_{a_0}^{a(t)} \frac{da}{\Delta - a(\Gamma + \Delta)} = \int_0^t 4dt$$
$$ln\frac{\Delta - a(\Gamma + \Delta)}{\Delta - a_0(\Gamma + \Delta)} = -4t(\Gamma + \Delta)$$
$$e^{4t(\Gamma+\Delta)} = \frac{\Delta - a(\Gamma + \Delta)}{\Delta - a_0(\Gamma + \Delta)}$$
$$\frac{\Delta - [\Delta - a(0)(\Gamma + \Delta)]e^{-4t(\Gamma+\Delta)}}{\Gamma + \Delta} = a(t)$$

Similarly,

$$\frac{db}{dt} = (\Gamma_1 + \Gamma_2)(-2b(t) + (\Delta_1 + \Delta_2)(-2b(t))$$
$$\int_{b(0)}^{b(t)} \frac{db}{b(t)} = \int_0^t -2(\Gamma + \Delta)dt$$
$$ln\frac{b(t)}{b(0)} = -2t(\Gamma + \Delta)$$
$$b(t) = b(0)e^{-2t(\Gamma+\Delta)}$$

Similarly, we get $b^*(t) = b^*(0)e^{-2(\Gamma+\Delta)}$ and $c(t) = 1 - a(t)$. This gives us the state of the system at a general time $t$, which is the action of the dynamical map.

## 3 Fixed states of the Dynamical Map

A fixed state is defined as a state $\rho_{\text{fixed}}$ such that $\Lambda(t)\rho_{\text{fixed}} = \rho_{\text{fixed}} \forall t$. Using this definition, and the equation for $\rho(t)$ calculated in the previous section, we can find the fixed states by solving for each term individually.

$$\Delta - [\Delta - a(0)(\Gamma + \Delta)]e^{-4t(\Gamma-\Delta)} = a(0)(\Gamma + \Delta)$$
$$\Delta(1 - e^{-4t(\Gamma-\Delta)} + a(0)(\Gamma + \Delta)e^{-4t(\Gamma-\Delta)} = a(0)(\Gamma + \Delta)$$
$$a(0) = \frac{\Delta(1 - e^{-4t(\Gamma-\Delta)}}{(\Gamma + \Delta)(1 - e^{-4t(\Gamma-\Delta)}}$$
$$a(0) = \frac{\Delta}{\Gamma + \Delta}$$

From the previous section, we have

$$b(t) = b(0)e^{-2t(\Gamma+\Delta)} = b(0)\ \forall t$$
$$\Rightarrow b(0) = 0$$
$$\text{Therefore, } b^*(0) = 0$$

Also,

$$\text{We know that } c(0) = 1 - a(0)$$
$$\therefore c(0) = \frac{\Gamma}{\Gamma + \Delta}$$

That gives us the following matrix

$$\rho_{fixed} = \begin{pmatrix} a(0) & b(0) \\ b^*(0) & c(0) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\Delta}{\Delta+\Gamma} & 0 \\ 0 & \frac{\Gamma}{\Delta+\Gamma} \end{pmatrix}$$

These initial values for the density matrix give us a state which is fixed under the given map.

## 4 Kraus Operators Corresponding to the map

If we are given a dynamical map $\Lambda_d$, acting on a $d$-dimensional quantum system, we have a simple alorithm to find he kraus operator.

1. Find the Choi matrix $C_d$ for this map as follows:

$$\mathbb{I}_d \otimes \Lambda_d(|\psi\rangle\langle\psi|) = C_d$$
$$\text{where, } |\psi\rangle = \frac{\sum_{i=1}^{d} |ii\rangle}{\sqrt{d}}$$

   where $|\psi\rangle$ is a maximally entangled state in $d$-dimensions.

2. Diagonalize $C_d$ as follows:

   $C_d = \sum_\alpha \lambda_\alpha |\alpha\rangle\langle\alpha|$, where $|\alpha\rangle$ is a $d \times d$ vector.

   Now let, $|\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d^2} \end{pmatrix}$,

   and matrix $A = \begin{pmatrix} a_1 & a_{d+1} & \dots & \\ a_2 & a_{d+2} & \dots & \\ \vdots & \vdots & & \\ a_d & a_{2d} & \dots & a_{d^2} \end{pmatrix}$

3. The $\alpha^{th}$ Kraus operator will be given by $K_\alpha = \sqrt{\lambda_\alpha} A_\alpha$

In our case, we know how the map acts on each element of the density matrix. We can use that to find the choi matrix directly. [1]

$$\begin{aligned} [\mathbb{I}_2 \otimes \Lambda](\rho) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [\Lambda](\rho) \\ &= \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix} \\ &= \begin{bmatrix} \Lambda\rho_1 & \Lambda\rho_2 \\ \Lambda\rho_3 & \Lambda\rho_4 \end{bmatrix} \end{aligned}$$

Where $\rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$

$$\begin{aligned} |\psi\rangle &= \frac{\sum_{i=1}^{d} |ii\rangle}{\sqrt{d}} \\ |\psi\rangle\langle\psi| &= \frac{\sum_{i=1}^{d}\sum_{j=1}^{d} |ii\rangle\langle jj|}{d} \\ \text{Substituting } d = 2, |\psi\rangle\langle\psi| &= \frac{1}{2}(|11\rangle\langle 11| + |11\rangle\langle 22| + |22\rangle\langle 11| + |22\rangle\langle 22|) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

[1] In the calculation of $\Lambda_t$ it was assumed that the diagonal elements sum to 1. That will not be the case here. We instead get a coupled differential equation. But solving that, we saw that the sum should be a constant. And therefore the form of the matrix will not change

Now,

$$C = (\mathbb{I}_2 \otimes \Lambda) |\psi\rangle \langle\psi| = \frac{1}{2} \begin{pmatrix} \frac{\Delta - [\Delta - \Gamma] e^{-4t(\Delta+\Gamma)}}{\Delta+\Gamma} & 0 & 0 & e^{-2t(\Delta+\Gamma)} \\ 0 & \frac{\Gamma}{\Gamma+\Delta} & 0 & 0 \\ 0 & 0 & \frac{\Delta}{\Gamma+\Delta} & 0 \\ e^{-2t(\Delta+\Gamma)} & 0 & 0 & \frac{\Gamma + (+\Gamma) \frac{e}{2}^{-4t(\Delta+\Gamma)}}{\Delta+\Gamma} \end{pmatrix}$$

$$C = \frac{1}{2} \begin{pmatrix} c_1 & 0 & 0 & c_2 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_4 & 0 \\ c_5 & 0 & 0 & c_6 \end{pmatrix}$$

The eigenvalue is given as:

$$\lambda_1 = c_3$$

$$\lambda_2 = c_4$$

$$\lambda_3 = \frac{1}{2} \left( c_1 + c_6 - \sqrt{c_1^2 - 2\, c_6\, c_1 + c_6^2 + 4\, c_2\, c_5} \right)$$

$$\lambda_4 = \frac{1}{2} \left( c_1 + c_6 + \sqrt{c_1^2 - 2\, c_6\, c_1 + c_6^2 + 4\, c_2\, c_5} \right)$$

Figure 2: Eigenvalue of the Choi matrix derived above

The eigenvector is given as:

$$v_1 = (0, 1, 0, 0)$$

$$v_2 = (0, 0, 1, 0)$$

$$v_3 = \left(-\frac{-c_1 + c_6 + \sqrt{c_1^2 + 4\,c_2\,c_5 - 2\,c_1\,c_6 + c_6^2}}{2\,c_5}, 0, 0, 1\right)$$

$$v_4 = \left(-\frac{-c_1 + c_6 - \sqrt{c_1^2 + 4\,c_2\,c_5 - 2\,c_1\,c_6 + c_6^2}}{2\,c_5}, 0, 0, 1\right)$$

Figure 3: Eigenvector of the Choi matrix derived above

Now that we know the eigenvalues and the eigenvector for the Choi matrix, we can follow step 3 of the algorithm stated above to find the specific kraus operator[2].

# 5 Entropy Production Rate

We need to find the following quantity called Entropy Production Rate:

$$\sigma(t) = -\frac{d}{dt} S(\rho(t)||\rho_{fix})$$
$$\text{where, } S(\rho_1||\rho_2) = Tr[\rho_1(\ln \rho_1 - \ln \rho_1]$$

Now,

$$\begin{aligned}
S(\rho_1||\rho_{fixed}) &= Tr[\rho_1(\ln(\rho_1) - \ln(\rho_{fixed})] \\
&= Tr(\rho_t \ln(\rho_t) - Tr(\rho_t \ln(\rho_{fixed})) \\
&= Tr(U\Lambda_t U^{-1} \ln\big(U\Lambda_t U^{-1}\big)) - Tr(\rho_t \ln(\rho_{fixed})) \\
&= Tr(U\Lambda_t U^{-1} U \ln(\Lambda_t) U^{-1}) - Tr(\rho_t \ln(\rho_{fixed})) \\
\therefore S(\rho_1||\rho_{fixed}) &= Tr(\Lambda_t \ln(\Lambda_t)) - Tr(\rho_t \ln(\rho_{fixed}))
\end{aligned}$$

Note that, $\Lambda_t$ is diagonal, since it is the eigenvalue matrix of $\rho_t$, and $\rho_{fixed}$ is already diagonal.
Recall that

$$\begin{aligned}
\rho(t) &= \begin{pmatrix} a(t) & b(t) \\ b^*(t) & c(t) \end{pmatrix}, \text{ where} \\
a(t) &= \frac{\Delta - [\Delta - a(0)(\Gamma + \Delta)]e^{-4t(\Gamma+\Delta)}}{\Gamma + \Delta} \\
b(t) &= b(0)e^{-2t(\Gamma+\Delta)} \\
b^*(t) &= b^*(0)e^{-2t(\Gamma+\Delta)} \\
c(t) &= 1 - a(t)
\end{aligned}$$

[2]The specific calculation here is ommitted since the operator will be used in a specific system-environment setting, which is a situation of known $\Gamma$ and $\Delta$, and therefore simple expressions for the eigenvalue and eigenvector can be derived

To find the eigenvalues of $\rho(t)$, consider the following:

$$|\rho(t)-\lambda| = \begin{vmatrix} x-\lambda & b(t) \\ b^*(t) & 1-x-\lambda \end{vmatrix} = 0, \text{ where x = a(t), and 1-x = c(t)}$$

$$\begin{aligned}
(x-\lambda)(1-x-\lambda) - |b(t)|^2 &= 0 \\
(x-\lambda) - (x^2-\lambda^2) - |b(t)|^2 &= 0 \\
x - \lambda - x^2 - \lambda^2 - |b(t)|^2 &= 0 \\
\lambda^2 - \lambda + x - x^2 - |b(t)|^2 &= 0
\end{aligned}$$

By solving the above equation, we get $\lambda = \frac{1 \pm \sqrt{1-4(x-x^2-|b(t)|^2}}{2}$
On substituting the values and further simplifying, we get the following eigenvalues:

$$\begin{aligned}
\lambda_1 \approx \frac{1}{a+b}\, & 0.5 \times 2.71828^{-4t(a+b)} \Big(-\sqrt{\big((a(-2.71828^{4t(a+b)}) - b\, 2.71828^{4t(a+b)} - a_0 b - a c_0 -} \\
& a_0 a + a - b c_0 + b)^2 - \\
& 4 (a_0 a^2\, 2.71828^{4t(a+b)} - a^2 b_0 b_1\, 2.71828^{4t(a+b)} + a_0 a^2 c_0 - a_0 a^2 + \\
& b^2 c_0\, 2.71828^{4t(a+b)} + a_0 b^2 c_0 - b^2 b_0 b_1\, 2.71828^{4t(a+b)} + \\
& a b c_0\, 2.71828^{4t(a+b)} - a b c_0 + 2 a_0 a b c_0 - 2 a b \\
& 2.71828^{4t(a+b)} + a b\, 2.71828^{8t(a+b)} + a_0 a b\, 2.71828^{4t(a+b)} - \\
& 2 a b b_0 b_1\, 2.71828^{4t(a+b)} + a b - a_0 a b - b^2 c_0)\big) + \\
& a\, 2.71828^{4t(a+b)} + b\, 2.71828^{4t(a+b)} + \\
& a_0 \\
& b + a \\
& c_0 + a_0 \\
& a - a + b \\
& c_0 - b\Big)
\end{aligned}$$

$$\begin{aligned}
\lambda_2 \approx \frac{1}{a+b}\, & 0.5 \times 2.71828^{-4t(a+b)} \\
& \Big(\sqrt{\big((a(-2.71828^{4t(a+b)}) - b\, 2.71828^{4t(a+b)} - a_0 b - a c_0 - a_0 a + a - b c_0 + b)^2 -} \\
& 4 (a_0 a^2\, 2.71828^{4t(a+b)} - a^2 b_0 b_1\, 2.71828^{4t(a+b)} + a_0 a^2 c_0 - a_0 a^2 + \\
& b^2 c_0\, 2.71828^{4t(a+b)} + a_0 b^2 c_0 - b^2 b_0 b_1\, 2.71828^{4t(a+b)} + \\
& a b c_0\, 2.71828^{4t(a+b)} - a b c_0 + 2 a_0 a b c_0 - 2 a b \\
& 2.71828^{4t(a+b)} + a b\, 2.71828^{8t(a+b)} + a_0 a b\, 2.71828^{4t(a+b)} - \\
& 2 a b b_0 b_1\, 2.71828^{4t(a+b)} + a b - a_0 a b - b^2 c_0)\big) + \\
& a\, 2.71828^{4t(a+b)} + b\, 2.71828^{4t(a+b)} + a_0 b + \\
& a c_0 + \\
& a_0 a - \\
& a + \\
& b c_0 - \\
& b\Big)
\end{aligned}$$

The eigenvectors corresponding to these eigenvalues were found to be:

$$v_1 \approx \left(-\frac{1}{(a+b)\,b_1} 0.5\times 2.71828^{-2t(a+b)} \left(-a + 2.71828^{4t(a+b)}\,a + b - 2.71828^{4t(a+b)}\,b - a\,a_0 - b\,a_0 + a\,c_0 + b\,c_0 + \sqrt{\left(a^2 - 2\times 2.71828^{4t(a+b)}\,a^2 + 2.71828^{8t(a+b)}\,a^2 - 2\,a\,b + 4\times 2.71828^{4t(a+b)}\,a\,b - 2\times 2.71828^{8t(a+b)}\,a\,b + b^2 - 2\times 2.71828^{4t(a+b)}\,b^2 + 2.71828^{8t(a+b)}\,b^2 + 2\,a^2\,a_0 - 2\times 2.71828^{4t(a+b)}\,a^2\,a_0 - 2\,b^2\,a_0 + 2\times 2.71828^{4t(a+b)}\,b^2\,a_0 + a^2\,a_0^2 + 2\,a\,b\,a_0^2 + b^2\,a_0^2 + 4\times 2.71828^{4t(a+b)}\,a^2\,b_0\,b_1 + 8\times 2.71828^{4t(a+b)}\,a\,b\,b_0\,b_1 + 4\times 2.71828^{4t(a+b)}\,b^2\,b_0\,b_1 - 2\,a^2\,c_0 + 2\times 2.71828^{4t(a+b)}\,a^2\,c_0 + 2\,b^2\,c_0 - 2\times 2.71828^{4t(a+b)}\,b^2\,c_0 - 2\,a^2\,a_0\,c_0 - 4\,a\,b\,a_0\,c_0 - 2\,b^2\,a_0\,c_0 + a^2\,c_0^2 + 2\,a\,b\,c_0^2 + b^2\,c_0^2\right)}\right), 1\right)$$

$$v_2 \approx \left(-\frac{1}{(a+b)\,b_1} 0.5\times 2.71828^{-2t(a+b)} \left(-a + 2.71828^{4t(a+b)}\,a + b - 2.71828^{4t(a+b)}\,b - a\,a_0 - b\,a_0 + a\,c_0 + b\,c_0 - \sqrt{\left(a^2 - 2\times 2.71828^{4t(a+b)}\,a^2 + 2.71828^{8t(a+b)}\,a^2 - 2\,a\,b + 4\times 2.71828^{4t(a+b)}\,a\,b - 2\times 2.71828^{8t(a+b)}\,a\,b + b^2 - 2\times 2.71828^{4t(a+b)}\,b^2 + 2.71828^{8t(a+b)}\,b^2 + 2\,a^2\,a_0 - 2\times 2.71828^{4t(a+b)}\,a^2\,a_0 - 2\,b^2\,a_0 + 2\times 2.71828^{4t(a+b)}\,b^2\,a_0 + a^2\,a_0^2 + 2\,a\,b\,a_0^2 + b^2\,a_0^2 + 4\times 2.71828^{4t(a+b)}\,a^2\,b_0\,b_1 + 8\times 2.71828^{4t(a+b)}\,a\,b\,b_0\,b_1 + 4\times 2.71828^{4t(a+b)}\,b^2\,b_0\,b_1 - 2\,a^2\,c_0 + 2\times 2.71828^{4t(a+b)}\,a^2\,c_0 + 2\,b^2\,c_0 - 2\times 2.71828^{4t(a+b)}\,b^2\,c_0 - 2\,a^2\,a_0\,c_0 - 4\,a\,b\,a_0\,c_0 - 2\,b^2\,a_0\,c_0 + a^2\,c_0^2 + 2\,a\,b\,c_0^2 + b^2\,c_0^2\right)}\right), 1\right)$$

Recall that

$$\begin{aligned}
\Lambda_t &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\
\therefore \Lambda_t \ln(\Lambda_t) &= \begin{pmatrix} \lambda_1 \ln(_1) & 0 \\ 0 & \lambda_2 \ln(\Lambda_2) \end{pmatrix} \\
\text{and } \rho &= \begin{pmatrix} \frac{\Delta}{\Delta+\Gamma} & 0 \\ 0 & \frac{\Gamma}{\Delta+\Gamma} \end{pmatrix} \\
\therefore \rho(t) \ln(\rho_{fixed}) &= \begin{pmatrix} a(t) \ln\left(\frac{\Delta}{\Delta+\Gamma}\right) & 0 \\ 0 & c(t) \ln\left(\frac{\Gamma}{\Delta+\Gamma}\right) \end{pmatrix}
\end{aligned}$$

Now,

$$\begin{aligned}
S(\rho_1 || \rho_{fixed}) &= Tr(\Lambda_t \ln(\Lambda_t)) - Tr(\rho_t \ln(\rho_{fixed})) \\
&= \lambda_1 \ln(\lambda_1) + \lambda_2 \ln(\lambda_2) - a(t) \ln\left(\frac{\Delta}{\Delta+\Gamma}\right) - c(t) \ln\left(\frac{\Gamma}{\Delta+\Gamma}\right)
\end{aligned}$$

On simplifying, we get:

$$S(\rho(t)||\rho_{fixed}) = \lambda_1 \ln(\lambda_1) + \lambda_2 \ln(\lambda_2) - a(t)(\ln\left(\frac{\Delta}{\Gamma}\right)) -$$
$$ln(\Gamma) + \ln(\Delta + \Gamma)$$
$$\implies \frac{dS(\rho_1||\rho_{fixed})}{dt} = \frac{d}{dt}(\lambda_1 \ln(\lambda_1) + \lambda_2 \ln(\lambda_2) - a(t)(\ln\left(\frac{\Delta}{\Gamma}\right)) -$$
$$ln(\Gamma) + \ln(\Delta + \Gamma))$$
$$= \ln(\lambda_1)\frac{d\lambda_1}{dt} + \frac{d\lambda_1}{dt} + \ln(\lambda_2)\frac{d\lambda_2}{dt} + \frac{d\lambda_1}{dt} - \ln\left(\frac{\Delta}{\Gamma}\right)\frac{da(t)}{dt}$$
$$= \ln(\lambda_1 + 1)\frac{d\lambda_1}{dt} + \ln(\lambda_2 + 1)\frac{d\lambda_2}{dt} - \ln\left(\frac{\Delta}{\Gamma}\right)\frac{da(t)}{dt}$$

Now,

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1 - 4(a - a^2 - |b|^2)}}{2}$$
$$\implies \frac{d\lambda_{\pm}}{dt} = \frac{\mp 2|b|^2(\Gamma + \Delta)(e^{-2t(\Delta+\Gamma)}) - \dot{a}}{\sqrt{1 - 4(a - a^2 - |b|^2)}}$$
$$\text{where } \dot{a} = \frac{da(t)}{dt} = 4[\Delta - a_0(\Gamma + \Delta)]e^{-4t(\Delta+\Gamma)}$$

This gives us the analytical expression for the entropy production rate. Let's see how it behaves when we assume the initial state to be a maximally mixed state or some values of $\Gamma$ and $\Delta$.

$$\rho(0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{7}$$

Set $\gamma_1 = \gamma_2 = n_1 = n_2 = 1$. Therefore,

$$a(t) = \frac{2 + e^{-24t}}{6} \tag{8}$$
$$c(t) = \frac{2 - e^{-24t}}{6} \tag{9}$$
$$b(t) = b^*(t) = 0 \tag{10}$$
$$\rho_{fix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \tag{11}$$

On simplifying for $\sigma(t)$, we get

$$\sigma(t) = -4e^{-24t} \ln \frac{2 - e^{-24t}}{2 + e^{-24t}} \tag{12}$$

On plotting the value of entropy production rate with time, we get the following graph:

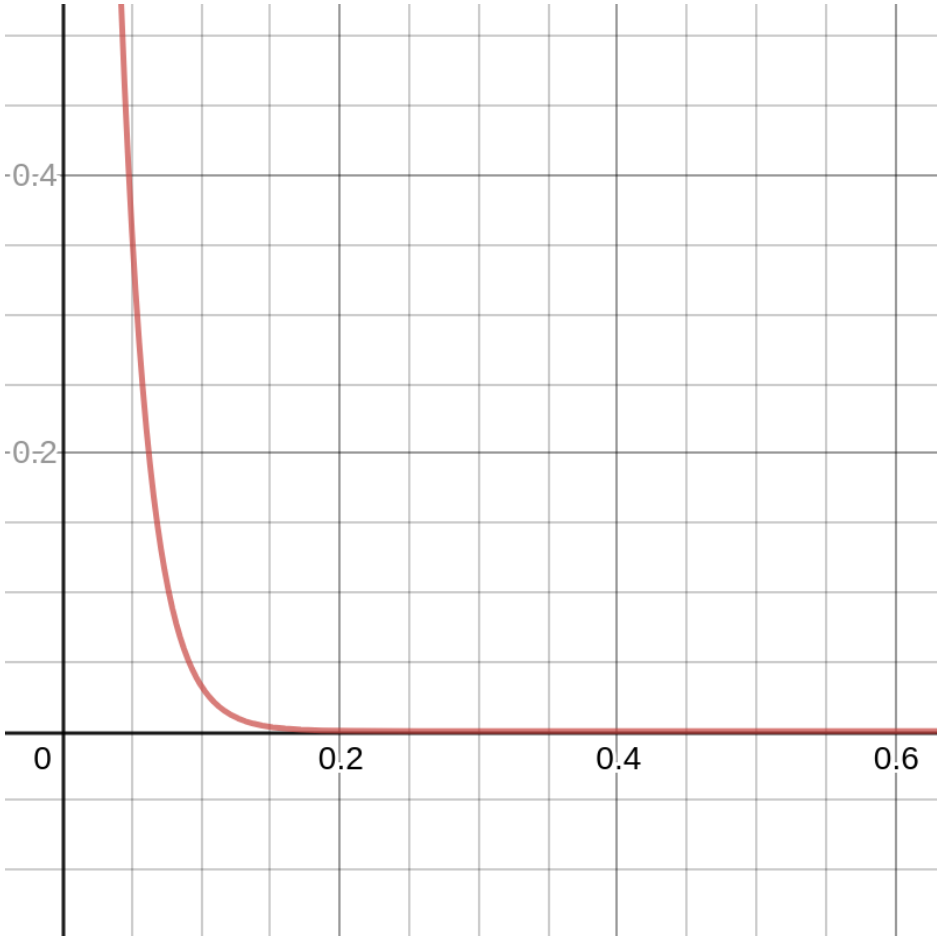


Figure 4: Entropy production rate for maximally mixed state

From the graph, it can be seen that the entropy production increases from 1.908 at $t = 0$ to 0 pretty quickly, and then stays close to 0. We can interpret $\gamma_i$ to be some sort of coupling constant between the system and the environment and $n_i$ to be a sort of "strength" of the environment. A higher value of $n$ should give more interesting plots for the quantity. We also want to study $\sigma(t)$ as a function of both $n$ and $\gamma$, instead of fixing them. We thought of setting $\gamma_2 = 0$ and using $\gamma_1 = \gamma; n_1 = n$ and then plotting the resultant expressions, but the time limit here did not permit the simplification required for that plot.

# 6 Ideas for extending this work

I want to understand how exactly $n$ and $\gamma$ are affecting the value of $\sigma(t)$ and how are they affecting the state of the system (specifically, how is the value of coherence changing under this equation.) I thought that would be interesting since it would give some insight to how to model a qubit in the context of gate application in a quanum circuit, by modelling the gate application as an environment and using appropriate function for $n$ and $\gamma$. The quantity that we want to specifically look at is *Quantum coherence*, which measures the degree of superposition in a quantum system. One possible quantisation for it is $l_1$ norm of coherence which for a density matrix $\rho$ is defined as follows.

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| \tag{13}$$

Although the practical scenario of multiple qubit interacion will be fairly more complicated, the behaviour of coherence would certainly be interesting to see.


# 7 Acknowledgement

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# References


[1] Angel Rivas and Susana F. Huelga. *Open Quantum Systems*. Springer Berlin Heidelberg, 2012. DOI: 10.1007/978-3-642-23354-8. URL: https://doi.org/10.1007/978-3-642-23354-8.

[2] Daniel Manzano. "A short introduction to the Lindblad master equation". In: *AIP Advances* 10.2 (2020), p. 025106. DOI: 10.1063/1.5115323.